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LETTER TO THE EDITOR

Identity of commutator and Poisson bracket

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Abstract. The commutator of two functions of n noncommuting variables is proved to be identical to their Fréchet–Poisson bracket.

Lagrange's equations in quantum theory involve derivatives of formal functions of operators with respect to these operators. Such derivatives are naturally defined in the following way. If f, g are functions of (not necessarily commuting) variables $q^1, \ldots, q^n$, define

$$\left(\frac{\partial f}{\partial q^i}, g\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(q^1, \ldots, q^{i-1}, q^i + \epsilon g, q^{i+1}, \ldots, q^n) - f(q)). \tag{1}$$

We denote by $\mathscr{A}$ the algebra of formal polynomials and power series in the variables $q^1, \ldots, q^n$, with complex coefficients. A *derivation D* on $\mathscr{A}$ is a linear map from $\mathscr{A}$ into $\mathscr{A}$ with the further property that

$$D(fg) = (Df)g + f(Dg).$$

Evidently if D_1 and D_2 are two derivatives on $\mathscr{A}$ such that $D_1 q^i = D_2 q^i$ for all q^i then $D_1 f = D_2 f$ for all f in $\mathscr{A}$. The mapping $\{f, g\} \to (\partial f/\partial q^i, g)$ is a bilinear map from $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$ and for fixed $g \in \mathscr{A}$ it is a derivation on $\mathscr{A}$, since

$$\left(\frac{\partial}{\partial q^i}(f_1 f_2), g\right) = f_1\left(\frac{\partial}{\partial q^i} f_2, g\right) + \left(\frac{\partial}{\partial q^i} f_1, g\right) f_2.$$

We shall call $\partial f/\partial q^i$ the Fréchet derivative of f with respects to q^i in loose analogy to the Fréchet derivative of a mapping between Banach spaces.

In this note we give a proof of the identity:

$$[f, g] \equiv \left(\frac{\partial f}{\partial q^i}, \left(\frac{\partial g}{\partial q^j}, [q^i, q^j]\right)\right). \tag{2}$$

(Repeated indices are summed.)

In the special case when the basis of $\mathscr{A}$ consists of n mutually commuting q's, and n mutually commuting p's, the equation (2) becomes

$$[f(q,p), g(q,p)] \equiv \left(\frac{\partial f}{\partial q^i}, \left(\frac{\partial g}{\partial p_j}, [q^i, p_j]\right)\right) - \left(\frac{\partial f}{\partial p_j}, \left(\frac{\partial g}{\partial q^i}, [q^i, p_j]\right)\right). \tag{3}$$

The right-hand side of equation (3) is the natural extension of the Poisson bracket to

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noncommuting operators, so we shall call the right-hand side of equation (2) the Fréchet–Poisson bracket. The equation (3) may be helpful in the problem of quantizing arbitrary classical systems.

We proceed to the proof of equation (2). We observe first that

$$\left(\frac{\partial f}{\partial q^i}, [g, q^i]\right) = [g, f] \tag{4}$$

since for fixed g, both sides are derivations on f which agree when $f = q^j$. Hence equation (4) holds for all f, $g \in \mathscr{A}$. Letting $g = q^j$ in equation (4) and replacing f by g yields

$$\left(\frac{\partial g}{\partial q^i}, [q^j, q^i]\right) = [q^j, g].$$

Then substituting the left-hand side for $[q^j, g]$ into equation (4) yields the desired equation (2).